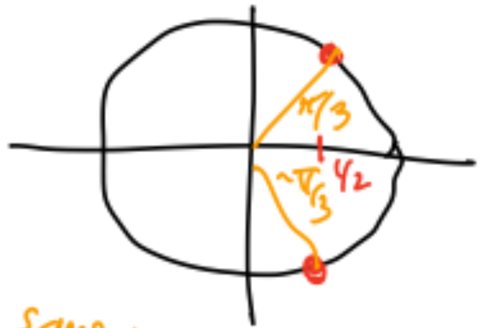
(4.31) Part of this: solve

$2\cos\left(2x-\frac{\pi}{3}\right)-1=0$ on $\left[-\pi,\frac{\pi}{2}\right]$

$\iff \cos(\underbrace{2x-\tfrac{\pi}{3}}_{\theta}) = \frac{1}{2}$



$\iff \theta = \frac{\pi}{3} + 2n\pi$ for some $n \in \mathbb{Z}$

or $\theta = -\frac{\pi}{3} + 2k\pi$ for some $k \in \mathbb{Z}$.

$\iff$ $2x - \frac{\pi}{3} = \frac{\pi}{3} + 2n\pi$ for some $n \in \mathbb{Z}$

$\iff 2x = \frac{2\pi}{3} + 2n\pi$

$\boxed{x = \frac{\pi}{3} + n\pi \text{ for some } n \in \mathbb{Z}}$

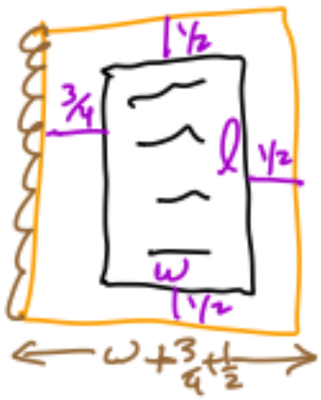
OR $2x - \frac{\pi}{3} = -\frac{\pi}{3} + 2k\pi$ for some $k \in \mathbb{Z}$

$\iff 2x = 2k\pi$ for some $k \in \mathbb{Z}$

$\iff \boxed{x = k\pi \text{ for some } k \in \mathbb{Z}}$

Given $x \in \left[-\pi, \frac{\pi}{2}\right]$ $x = -\pi, -\frac{2\pi}{3}, 0,$ or $\frac{\pi}{3}$.

Example A publishing company designs a book with pages that are 80 square inches. Each page needs a $\frac{3}{4}$" margin on the binding side and $\frac{1}{2}$" margins on the other 3 sides. How should the pages be designed so that the printed area (with the text) is as large as possible?



Given in^2

$$80 = \left(w + \tfrac{1}{2} + \tfrac{3}{4}\right)\cdot\left(l + \tfrac{1}{2} + \tfrac{1}{2}\right)$$

$$\boxed{80 = \left(w + \tfrac{5}{4}\right)(l+1)}$$

Want maximum of printed area $= A$ ← our function.

$$A = lw$$

$$\frac{80}{l+1} = w + \frac{5}{4}$$

$$w = \frac{80}{l+1} - \frac{5}{4}$$

$$\Rightarrow A(l) = l\cdot\left(\frac{80}{l+1} - \frac{5}{4}\right)$$

$$\Rightarrow A(l) = \frac{80l}{l+1} - \frac{5}{4}l.$$ maximize $A(l)$

What is the domain of l? $0" \le l \le 63"$

$$l+1 = \frac{80}{w+\frac{5}{4}}$$

$$l = \frac{80}{w+\frac{5}{4}} - 1 \Rightarrow$$

Math: maximize

$$A(\ell) = \frac{80\ell}{\ell+1} - \frac{5}{4}\ell$$

on $[0, 63]$ ← it will have a global max → at either an endpt or a cr pt.

$$\ell \le \frac{80}{0+\frac{5}{4}} - 1 = 80 \cdot \frac{4}{5} - 1 = 16 \cdot 4 - 1 = 64 - 1 = 63$$

Cr pts:

$$A'(\ell) = \frac{(\ell+1)80 - 80\ell}{(\ell+1)^2} - \frac{5}{4} = \frac{80}{(\ell+1)^2} - \frac{5}{4} = 0$$

$\ell = -1$ undefined.

$$\frac{80}{(\ell+1)^2} = \frac{5}{4} \qquad \frac{80 \cdot 4}{5} = (\ell+1)^2$$

$$64 = (\ell+1)^2$$

$$\Rightarrow (\ell+1) = \pm\sqrt{64} = \pm 8$$

$$\ell = 8-1 \text{ or } -8-1$$

$$\ell = 7 \text{ or } -9.$$

$\Rightarrow$ cr pts are $\ell = -1$, $\ell = 7$, $\ell = -9$

↑ only one in $[0, 63]$.

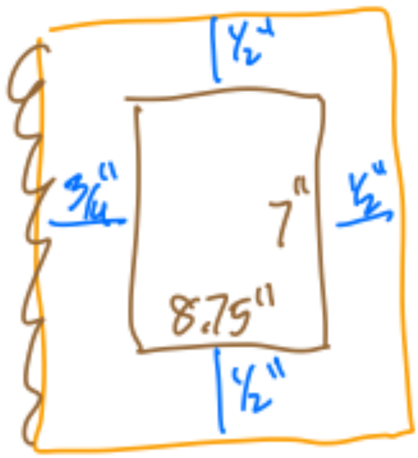
	ℓ	$A(\ell) = \frac{80\ell}{\ell+1} - \frac{5}{4}\ell$
cr pt	7	$\frac{80 \cdot 7}{8} - \frac{5}{4} \cdot 7 = 70 - \frac{35}{4} = \frac{245}{4} \approx 61.25 \text{ in}^2$
	0	0 in^2
	63	$\frac{80 \cdot 63}{64} - \frac{5}{4} \cdot 63 = \frac{\cancel{10}^{5} \cdot 63}{\cancel{8}4} - \frac{5 \cdot 63}{4} = 0$

$\boxed{\ell = 7 \text{ is global max}}$

To answer the question, we also need ω.

$$\omega = \frac{80}{\ell+1} - \frac{5}{4} = \frac{80}{8} - \frac{5}{4} = 10 - \frac{5}{4} = \frac{35}{4}$$
$$= 8.75''$$

$\therefore$ Best design



Another application of the derivative:

$\boxed{\text{l'Hôpital's Rule}}$

Review linearization of a function near a point x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

equation of the tangent line.

good approximation for x close to x_0.

(Check: $f(x_0) = f(x_0) + f'(x_0)(x_0 - x_0) = f(x_0)$. ✓
Plug in x_0.
Derivative $f'(x) \approx f'(x_0)$ ✓)

l'Hopital's Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$,

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, as long as both f & g are differentiable near a.

Proof: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + f'(a)\cdot(x-a)}{g(a) + g'(a)(x-a)}$

replacing each function by its linearization.

If $f(a) = g(a) = 0$,

$$= \lim_{x \to a} \frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}} = \frac{f'(a)}{g'(a)}.$$

works as long as the derivative of g is nonzero.

In all cases, if $f(a) = g(a) = 0$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \checkmark$$

Comment: The same l'Hopital's Rule works if the limit has the form $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

Example: Find $\lim\limits_{x\to 0} \dfrac{x - x^2e^x}{\arctan(3x)}$.

$\frac{0}{0}$ form. ✓ $\Rightarrow$ can use l'Hopital Rule.

$$\underset{l'H}{=} \lim_{x\to 0} \frac{1-(2xe^x + x^2e^x)}{\frac{1}{1+(3x)^2}\cdot 3} = \frac{1}{\frac{1}{1+0}\cdot 3} = \boxed{\frac{1}{3}}.$$

Note: We can also use l'Hopital's Rule in other indeterminant form situations:

$\boxed{\frac{0}{0}, \pm\frac{\infty}{\infty} \longrightarrow l'H}$

$0\cdot\infty \longrightarrow \frac{\infty}{1/0}$ or $\frac{0}{1/\infty}$ convert to using l'H.

$1^\infty, 0^0, \infty^0 \rightarrow$ let L be the limit, take log of both sides.

$\boxed{\text{Ex}}$ Find $\lim\limits_{x\to\infty} \underbrace{\sin\left(\tfrac{1}{x}\right)}_{0} \cdot \underbrace{e^x}_{\infty}$. indeterminate

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{1/e^x} = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{e^{-x}}$$

Now $\frac{0}{0}$ form

$$\underset{l'H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot \frac{-1}{x^2}}{-e^{-x}}$$

$$= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)}{x^2 e^{-x}} \quad \to 1$$

$$= \lim_{x\to\infty} \frac{1}{x^2 e^{-x}} = \lim_{x\to\infty} \frac{e^x}{x^2}$$

$\frac{\infty}{\infty}$

$$\underset{l'H}{=} \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \boxed{\infty}$$

$\frac{\infty}{\infty}$

Ex Find $\displaystyle\lim_{x\to\infty} \frac{x^5 - 2x^7 + 348}{x^4 + 3x^7 - 12}$

Method 1: $\displaystyle\lim_{x\to\infty} \frac{\cancel{x^7}\left(\frac{1}{x^2} - 2 + \frac{348}{x^7}\right)}{\cancel{x^7}\left(\frac{1}{x^3} + 3 - \frac{12}{x^7}\right)}$ $\to 0$, $\to 0$, $\to 0$, $\to 0$

$$= \boxed{\frac{-2}{3}}.$$

Method 2: $\displaystyle \lim_{x\to\infty} \frac{x^5 - 2x^7 + 348}{x^4 + 3x^7 - 12}$

form $\dfrac{-\infty}{+\infty}$

$$\underset{l'H}{=} \lim_{x\to\infty} \frac{5x^4 - 14x^6}{4x^3 + 21x^6}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^4}^{\,x}(5 - 14x^2)}{\cancel{x^3}(4 + 21x^3)}$$

$$= \lim_{x\to\infty} \frac{5x - 14x^3}{4 + 21x^3}$$ $\dfrac{-\infty}{\infty}$ form

$$\underset{l'H}{=} \lim_{x\to\infty} \frac{5 - 42x^2}{63x^2}$$ $\dfrac{-\infty}{\infty}$ form

$$= \lim_{x\to\infty} \frac{-84\cancel{x}}{126\cancel{x}} = \frac{-84}{126}$$

$$= \frac{-\cancel{42}\cdot 2}{\cancel{42}\cdot 3} = \boxed{\frac{-2}{3}}.$$

$\boxed{Ex}$ $\lim_{x\to 0} (\cos(x))^{1/x^2}$

1^{∞} form.

Let $L = \lim_{x\to 0} (\cos(x))^{1/x^2}$

Take ln of both

$\ln(L) = \ln\left(\lim_{x\to 0} (\cos(x))^{1/x^2}\right)$

$\Rightarrow \ln(L) = \lim_{x\to 0} \ln\left((\cos(x))^{1/x^2}\right)$

because ln is a continuous fcn, we can switch the limit & the ln.

Exponent Rule for ln: $\ln(A^B) = B \cdot \ln(A)$

$\Rightarrow \ln(L) = \lim_{x\to 0} \frac{1}{x^2} \ln(\cos(x))$

$\frac{0}{0}$ form.

$\underset{l'H}{=} \lim_{x\to 0} \frac{(\ln(\cos x))'}{2x}$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos(x)}\cdot(-\sin(x))}{2x}$$

$$= \lim_{x\to 0} \frac{-\sin(x)}{2x\cdot\cos(x)}$$

$\to 1 \quad \to 1$

$$\boxed{\ln L} = \boxed{-\frac{1}{2}}.$$

$$\Rightarrow \boxed{L = e^{-1/2}}.$$

Quiz

① What is your name (with the letters mixed up.)?

② Favorite Halloween/other costume (current or past).

③ Find the derivative

ⓐ $\arctan(2x)$

ⓑ $\ln(\cos(x))$

ⓒ $x\cdot e^{x^2}$